On information flow and feedback in relay networks

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***Abstract*—We consider wireless relay networks where a source node communicates to a destination node with the help of multiple intermediate relay nodes. In wireless, if a node can send information to another node, typically it can also receive information from that node. Therefore, there are inherently a lot of opportunities for feeding back information in such networks. However, transmissions are not isolated and usually subject to broadcast and interference.**

In this paper, we ask the following question: Can the information transfer in both directions of a link be critical to maximizing the end-to-end communication rate in such networks? Equivalently, could one of the directions in each bidirected link (and more generally at least one of the links forming a cycle) be shut down and the capacity of the network still be approximately maintained? Our main result is to show that in any arbitrary Gaussian relay network with bidirected edges and cycles, we can always identify a directed acyclic subnetwork that approximately maintains the capacity of the original network. The edges of this subnetwork can be identified as the information carrying links, and the remaining links as feedback, which can only provide limited contribution to capacity.

I. Introduction

Feedback has been studied extensively for single-hop communication channels. Shannon showed that feedback cannot increase the capacity of the point-to-point discrete memoryless channel [1]. For the multiple access (MAC), broadcast and relay channels, feedback can potentially increase the capacity, but only through a power gain [2], [3]. More recently, it has been shown in [4] that feedback can provide unbounded gain in interference channels. In the recent years, there has been significant interest in larger networks where communication between nodes is established in multiple hops [5], [6], [7]. However, the study of the usefulness of feedback has been mostly limited to the above single-hop settings of a few nodes.

In this paper, we consider general relay networks where a source node communicates to a destination node with the help of multiple intermediate relay nodes. We are motivated by the observation that in wireless, nodes in a network are typically capable of both sending and receiving information, thus communication links between pairs of nodes are often bidirectional. That is, if a given node can send information to another node, it can also receive information from that node. Therefore, there is inherently a lot of "feedback" in wireless networks, though the nature of the feedback is significantly different from the idealized feedback models considered in single-hop settings. First, transmissions, and therefore also feedback, are not isolated in wireless and often subject to broadcast and superposition; each receiver observes a mixture

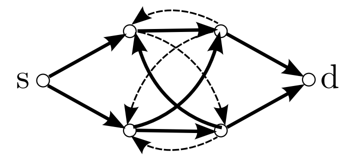

Fig. 1. Bidirected network with the directed acyclic subnetwork carrying approximately its capacity highlighted.

of the transmitted signals. Second, while in single hop networks the links originating from destinations and/or arriving at source nodes can be clearly distinguished as feedback, in multihop networks there can be "feedback" between any pair nodes. Bidirected links and cycles in the network can inherently feedback information, however it is not a priori possible to designate links as communication links and feedback. Therefore, in these new multi-hop settings it is not totally clear how to think about feedback and how to study its usefulness.

In this paper, we adopt the following approach. We consider a general Gaussian relay network with arbitrary topology and channel gains, possibly with bidirected links and cycles, where some links can be subject to broadcast and superposition and some can be isolated in a completely arbitrary fashion. We ask the following question: can the information transfer in both directions of a link be critical to maximizing the end-to-end communication rate in this network? Equivalently, could one of the directions in a bidirected link (and more generally at least one of the links forming a cycle) be shut down and the capacity of the network still be approximately maintained?

We show that in every wireless network with bidirected edges and cycles we can identify a directed acyclic subnetwork that approximately preserves the capacity of the original network. More precisely, if any of the links that do not belong to the identified subgraph could be disabled, the capacity of the resultant wireless network would still remain within a bounded gap to the capacity of the original network. The main technical step is to show that in every Gaussian relay network, there exists a directed acyclic subnetwork for which the information theoretic cutset upper bound evaluated under i.i.d. input distributions is exactly the same as that for the original network. See Figure 1.

Conceptually, this result identifies certain links in the network as the information carrying links (critical for information transfer) and the remaining as feedback (limited contribution to capacity). By identifying a directed acyclic subnetwork that approximately carries the whole capacity, it allows one to

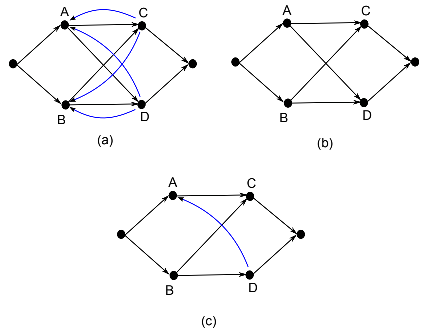

Fig. 2. Bidirected network with some of the links removed

associate a direction with the information flow in an undirected wireless network. Understanding which links are critical to maintaining the capacity of the network can be also useful for reducing the delay and complexity of the communication schemes by suggesting which links could be shut down. As studied in [8] and [9], cycles significantly increase the complexity of (approximately) optimal relaying strategies. However, in wireless networks it may not be always possible to disable individual links since these links may represent overheard transmissions over other links. Certain networks, such as Gaussian networks consisting of only MAC and broadcast components as studied by [7] and [10] provide some freedom in controlling individual links. However, simplification can be possible even in more general networks.

Consider the example in Fig.2-(a) where the edges in the graph indicate the wireless links with non-zero channel gains. Assume that the backward links from the second layer of relays (nodes C and D) to the first (nodes A and B) operate over a separate frequency, so that while signals arriving over the same colored edges superpose at a node, signals over different color edges arrive separately. Similarly, while signals over the same colored edges emanating from a single node represent broadcast, different signals can be transmitted over different colored edges. If the directed acyclic network in Fig.2-(b) is identified as sufficient for preserving the capacity of the network, this implies that the backward channel from the second layer to the first need not be used at all. On the other hand, if the directed subnetwork is the one in (c), there is no operational way to reduce the wireless network in (a) to (c). The forward link from node A to D cannot be avoided. However, the communication over the backward channel can still be simplified by not transmitting from node C and by avoiding the received signal at B.

II. Model

We consider a bidirected Gaussian relay network G consisting of a set of nodes V and communication links E. We let $|V|$ denote the total number of nodes. A source node $s \in V$ wants to communicate to a destination node $t \in V$. All nodes in the network are able to send and receive, thus, for each pair of nodes $u, v \in V$ we can potentially have links $(u, v) \in E$ and $(v, u) \in E$ with arbitrary channel gains. We assume the links with non-zero channel gains are represented with directed edges as in Fig.2-(a) giving rise to a directed graph with potentially bidirected edges and cycles. We assume nodes can have multiple transmit and receive antennas. Let $X_v \in \mathbb{C}^{M_v}$ denote the signal transmitted by node $v \in V$ with M_v transmit antennas. Similarly, let $Y_v \in \mathbb{C}^{N_v}$ denote the signal received by node $v \in V$ with N_v receive antennas. We have

$$Y_v = \sum_{u \in V} H_{vu} X_u + Z_v,$$

where H_{uv} denotes the channel matrix from node u to node v. This multiple-input multiple-output channel model can also be used to incorporate networks where different channels operate on different frequencies such as in Fig.2-(a), as well as isolated links.[1] The noise Z_v are independent and circularly symmetric Gaussian random vectors $\mathcal{N}(0, I)$. All nodes are subject to an average power constraint P. Note that the equal power constraint assumption is without loss of generality as the channel coefficients are arbitrary.

III. Main Result

For an arbitrary bidirected Gaussian relay network G with a set of nodes V and communication links E, we define a directed acyclic subnetwork $\tilde{G}$ to be one which consists of the same set of nodes V and a subset of the communication links $\tilde{E} \subseteq E$. For the Gaussian relay network, this corresponds to setting the channel coefficients corresponding to the edges in $E \setminus \tilde{E}$ to zero. A directed acyclic subnetwork satisfies the property that for any pair of nodes $u, v \in V$, if $(u, v) \in \tilde{E}$ then $(v, u) \notin \tilde{E}$. In other words, if there is a link in one direction between any two nodes, there cannot be a link in the opposite direction. Moreover, it contains no cycles. That is, for every set of nodes $v_1, \ldots, v_N \in V$, at least one of the edges $(v_1, v_2), (v_k, v_{k+1}), \ldots, (v_N, v_1) \notin \tilde{E}$ for any value of N.

The main conclusion of this paper is summarized in the following theorem.

Theorem 3.1: In any Gaussian relay network G with capacity $C(G)$, we can identify a directed acyclic subnetwork $\tilde{G}$ whose capacity $C(\tilde{G})$ in bits/s/Hz is bounded by

$$C(G) - g \leq C(\tilde{G}) \leq C(G) + g$$

where $g = 2\sum_{v \in V} M_v + (2 + \log 2)\sum_{v \in V} N_v$.

IV. Background

In [6], it has been shown that the capacity of any Gaussian relay network of the form defined in Section II is within a bounded gap to the information-theoretic cutset upper bound evaluated under i.i.d. Gaussian input distributions. We recall the following result from [11]:

Theorem 4.1 (Theorem 2.1, [11]): In any Gaussian relay network G, we can achieve all rates

$$R \leq \min_S I(X_S; Y_{S^c} | X_{S^c}) - g_1 \quad (1)$$

[1]Indeed, the conclusions of the paper also hold for wired networks (in this case with no gap) and a mixture of wireless and wired networks.

between s and d, where $g_1 = (2 + \log 2) \sum_{v \in V} N_v$, $S \subset V : s \in S, d \notin S$ is a source-destination cut of the network and $X_S = \{X_v,\, v \in S\}$ are i.i.d. $\mathcal{CN}(0, (P/M_v)I)$.

It has been shown in [6] that the restriction to i.i.d. Gaussian input distributions is within $g_2 = 2 \sum_{v \in V} M_v$ bits/s/Hz of the information-theoretic cut-set upper bound. This shows that within this total gap $g_1 + g_2$, the capacity of the network is approximately given by

$$C_{i.i.d.}(G) = \min_S f(G, S), \tag{2}$$

where

$$f(G; S) = I(X_S; Y_{S^c} | X_{S^c}), \tag{3}$$

and X_v, $v \in V$ are i.i.d. $\mathcal{CN}(0, (P/M_v)I)$. More precisely,

$$C_{i.i.d.}(G) - g_1 \leq C(G) \leq C_{i.i.d.}(G) + g_2. \tag{4}$$

Let us recall the definition of a submodular function: A submodular function $f : 2^V \to \mathbb{R}$ is defined as a function over subsets of V with diminishing returns, i.e., $\forall S_1, S_2 \subseteq V$,

$$f(S_1 \cap S_2) + f(S_1 \cup S_2) \leq f(S_1) + f(S_2).$$

The following theorem is proved in [12].

Theorem 4.2 (Theorem 1, [12]): For any G, $f(G; S) : 2^V \to \mathbb{R}$ as defined in (3) is a submodular function.

V. Proof of Main Result

The proof of the main result of this paper follows by combining the fact that the capacity of any Gaussian relay network is approximately given by $C_{i.i.d.}(G)$ in (4) and the following proposition.

Proposition 5.1: In every bidirected network G with $C_{i.i.d.}(G)$ given in (2), we can identify a directed acyclic subnetwork $\tilde{G}$ with $C_{i.i.d.}(\tilde{G}) = C_{i.i.d.}(G)$.

In the rest of this section, we concentrate on proving the proposition. We divide our proof into two parts. In the first part of the proof, we will show that for any pair of links (u, v) and (v, u), we can remove one of the links without changing $C_{i.i.d.}$. Given this new network, we can iterate this procedure for each bidirected link until we are left with a directed network that contains no bidirected edges. In the second part of the proof we show that given a directed network with cycles, we can remove at least one of the links in the cycle without changing $C_{i.i.d.}$. Iterating this procedure for each cycle, we can obtain a directed subnetwork of the same $C_{i.i.d.}$ that contains no cycles.

We note two important properties of the cut function:

1) For a fixed cut $S \subset V$, the cut values of a network G and subnetwork G' are the same if all outgoing links from S are in both G and G':
$f(G, S) = f(G', S)$, if $\forall (u, v) \in G : (u \in S, v \notin S), (u, v) \in G'$. (Note that because G' is a subgraph of G, the channel coefficients corresponding to (u, v) are the same in both G and G' if $(u, v) \in G'$.)
2) $f(G, S)$ is a submodular function on 2^V: $\forall S_1, S_2 \subseteq V$ $f(G, S_1) + f(G, S_2) \geq f(G, S_1 \cup S_2) + f(G, S_1 \cap S_2)$.

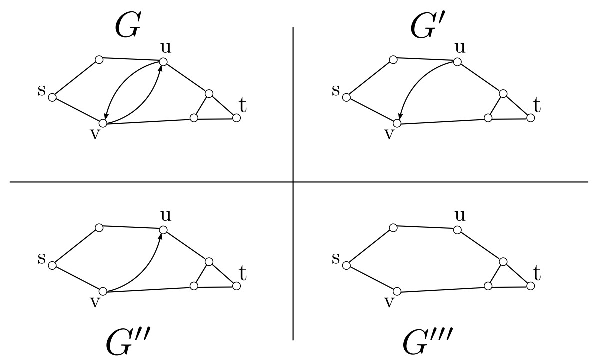

Fig. 3. Bidirected network with some of the links removed

The first property follows from the fact that when all outgoing links are in both G and G', the MIMO matrix between X_S and $Y_{\overline{S}}$ are the same, and thus $I(X_S; Y_{\overline{S}} | X_{\overline{S}})$ which corresponds to the capacity of this MIMO matrix is the same for both networks. The second property follows from Theorem 4.2.

A. Reduction of bidirected network to directed network

Given a bidirected network G and any pair of links (u, v) and (v, u), we create the subnetworks G', G'', and G''', where the link (v, u), (u, v), and both (u, v) and (v, u) are removed from G, respectively. See Figure 3.

Define S_v, S_u, S_{uv}, $S_{\overline{uv}}$, to be the following:

$$\begin{aligned} S_v &= \operatorname*{arg\,min}_{\{S: s, v \in S \;\; t, u \notin S\}} f(G, S) \\ S_u &= \operatorname*{arg\,min}_{\{S: s, u \in S \;\; t, v \notin S\}} f(G, S) \\ S_{uv} &= \operatorname*{arg\,min}_{\{S: s, u, v \in S \;\; t \notin S\}} f(G, S) \\ S_{\overline{uv}} &= \operatorname*{arg\,min}_{\{S: s \in S \;\; t, u, v \notin S\}} f(G, S). \end{aligned}$$

S_v is the cut with the minimum cut value among all cuts for which v remains on the source side and u remains on the destination side; S_u is the cut with the minimum cut value among all cuts for which u remains on the source side and v remains on the destination side; S_{uv} is the cut with the minimum cut value among all cuts for which both u and v are on the source side; and $S_{\overline{uv}}$ is the cut with the minimum cut value among all cuts for which u and v remain on the destination side. See Figure 4. A cut that achieves the minimum cut value need not be unique; we choose an arbitrary one in such cases. Note that

$$C_{i.i.d.}(G) = \min\left(f(G, S_v), f(G, S_u), f(G, S_{uv}), f(G, S_{\overline{uv}})\right). \tag{5}$$

We also define S'_v, S'_u, S'_{uv}, $S'_{\overline{uv}}$, and S''_v, S''_u, S''_{uv}, $S''_{\overline{uv}}$ in a similar fashion for graphs G' and G'', respectively.

Proposition 5.1 claims that either $C_{i.i.d.}(G) = C_{i.i.d.}(G')$ or $C_{i.i.d.}(G) = C_{i.i.d.}(G'')$. We prove this by showing that each of the following assumptions lead to a contradiction:

(a) $C_{i.i.d.}(G) < C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) < C_{i.i.d.}(G'')$;

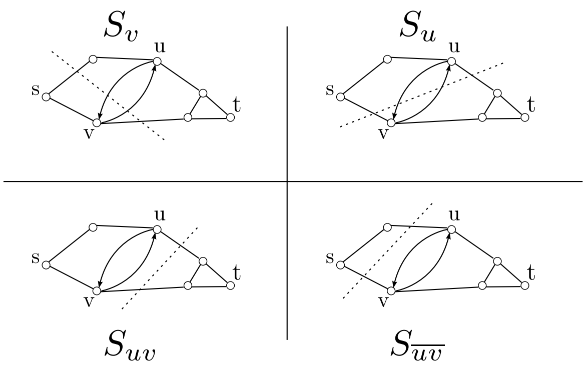

Fig. 4. Example of S_v, S_u, S_{uv}, $S_{\overline{uv}}$

(b) $C_{i.i.d.}(G) < C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) > C_{i.i.d.}(G'')$ (or equivalently, $C_{i.i.d.}(G) > C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) < C_{i.i.d.}(G'')$);
(c) $C_{i.i.d.}(G) > C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) > C_{i.i.d.}(G'')$.

Case (a): Assume $C_{i.i.d.}(G) < C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) < C_{i.i.d.}(G'')$. If $C_{i.i.d.}(G) < C_{i.i.d.}(G')$, then

$$C_{i.i.d.}(G) = f(G, S_v),$$

and

$$f(G, S_v) < \min\left(f(G, S_u), f(G, S_{uv}), f(G, S_{\overline{uv}})\right). \tag{6}$$

This can be seen as follows. Note that the minimums in the definitions of S_u, S_{uv} and $S_{\overline{uv}}$ are taken over a set of cuts that cannot cross the link (v, u) and G and G' only differ by the existence of the link (v, u). By Property (1), any cut that does not cross the edge (v, u) has the same value in G and G'. Therefore, $f(G, S_u) = f(G', S_u)$, $f(G, S_{uv}) = f(G', S_{uv})$ and $f(G, S_{\overline{uv}}) = f(G', S_{\overline{uv}})$. Now, if the minimum in (5) were to be achieved by any term other than $f(G, S_v)$, this would imply that $C_{i.i.d.}(G') \leq C_{i.i.d.}(G)$, which would contradict the assumption that $C_{i.i.d.}(G) < C_{i.i.d.}(G')$. Therefore, we have (6).

Now, if also $C_{i.i.d.}(G) < C_{i.i.d.}(G'')$, by the same argument above we should have

$$C_{i.i.d.}(G) = f(G, S_u),$$

and

$$f(G, S_u) < \min\left(f(G, S_v), f(G, S_{uv}), f(G, S_{\overline{uv}})\right). \tag{7}$$

But (6) and (7) are contradictory.

Case (b): Assume $C_{i.i.d.}(G) < C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) > C_{i.i.d.}(G'')$. Then, by the same argument in case (a), we have

$$C_{i.i.d.}(G) = f(G, S_v),$$

$$f(G, S_v) < \min\left(f(G, S_u), f(G, S_{uv}), f(G, S_{\overline{uv}})\right). \tag{8}$$

Similarly, the assumption $C_{i.i.d.}(G) > C_{i.i.d.}(G'')$ implies that

$$C_{i.i.d.}(G'') = f(G'', S''_u),$$

and

$$f(G'', S''_u) < \min\left(f(G'', S''_v), f(G'', S''_{uv}), f(G'', S''_{\overline{uv}})\right). \tag{9}$$

This follows by the same argument for (6): Since G and G'' only differ by the existence of (u, v), the value of the cut S_u should be different in G and G'' while the values of the remaining three cuts are the same in both G and G''.

Note that (8) implies that

$$f(G, S_v) < f(G, S_{\overline{uv}}) \leq f(G, S_v \cap S''_u),$$

where the last inequality follows from the fact that since $u \notin S_v$ and $v \notin S''_u$, $u, v \notin S_v \cap S''_u$. Now, by Property (1), $f(G, S_v) = f(G'', S_v)$ and $f(G, S_v \cap S''_u) = f(G'', S_v \cap S''_u)$ since G and G'' only differ by the existence of the link (u, v) and both S_v and $S_v \cap S''_u$ correspond to cuts that cannot cross this link. Therefore, we have

$$f(G'', S_v) < f(G'', S_v \cap S''_u).$$

On the other hand, (9) implies that

$$f(G'', S''_u) < f(G'', S''_{uv}) \leq f(G'', S_v \cup S''_u),$$

since $v \in S_v$ and $u \in S''_u$, $u, v \in S_v \cup S''_u$. Combining the last two inequalities we obtain

$$f(G'', S_v) + f(G'', S''_u) < f(G'', S_v \cap S''_u) + f(G'', S_v \cup S''_u).$$

However, submodularity (Property (2)) for f implies that

$$f(G'', S_v) + f(G'', S''_u) \geq f(G'', S_v \cap S''_u) + f(G'', S_v \cup S''_u),$$

which leads to a contradiction.

Case (c): Finally, we assume $C_{i.i.d.}(G) > C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) > C_{i.i.d.}(G'')$. By similar arguments as in case (b), the first assumption implies that $C_{i.i.d.}(G') = f(G', S'_v)$, and the second one implies that $C_{i.i.d.}(G'') = f(G'', S''_u)$. Moreover,

$$f(G', S'_v) < f(G, S'_v \cup S''_u), \tag{10}$$

and

$$f(G'', S''_u) < f(G, S'_v \cap S''_u). \tag{11}$$

The last two inequalities follow from our assumption, $C_{i.i.d.}(G) > C_{i.i.d.}(G')$ and $C_{i.i.d.}(G) > C_{i.i.d.}(G'')$, which implies that the minimum cut values for G' and G'' are strictly less than any cut value of G. Combining (10) and (11), we have

$$f(G', S'_v) + f(G'', S''_u) < f(G, S'_v \cap S''_u) + f(G, S'_v \cup S''_u),$$

Observing that by Property (1)

$$f(G', S'_v) = f(G''', S'_v) \tag{12}$$
$$f(G'', S''_u) = f(G''', S''_u) \tag{13}$$
$$f(G, S'_v \cap S''_u) = f(G''', S'_v \cap S''_u) \tag{14}$$
$$f(G, S'_v \cup S''_u) = f(G''', S'_v \cup S''_u), \tag{15}$$

we get

$$f(G''', S'_v)+f(G''', S''_u) < f(G''', S'_v\cap S''_u)+f(G''', S'_v\cup S''_u).$$

This contradicts with the submodularity of f in G'''.

Since we have eliminated cases (a), (b) and (c) above, we conclude that either $C_{i.i.d.}(G) = C_{i.i.d.}(G')$ or $C_{i.i.d.}(G) = C_{i.i.d.}(G'')$.

B. Removing cycles in a directed network

Consider a directed network G, where the nodes $\{v_1, v_2, \ldots v_N\}$ form a length N cycle, and let $v_{N+1} = v_1$. Define G_k, $k = 1, 2, \ldots, N$ to be a subnetwork of G with the link (v_k, v_{k+1}) removed. In our proof, we denote subnetworks with both links (v_{k-1}, v_k) and (v_k, v_{k+1}) removed as $G_{k-1,k}$. See Figure (5) for an example. Let S^* and S_k denote cuts that achieve the minimum cut values of networks G and G_k, respectively:

$$S^* = \underset{\{S: s\in S\ t\notin S\}}{\arg\min}\ f(G, S),$$
$$S_k = \underset{\{S: s\in S\ t\notin S\}}{\arg\min}\ f(G_k, S).$$

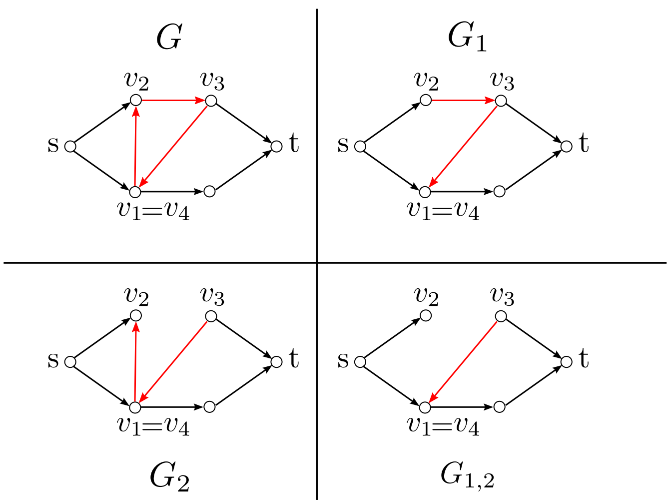

Fig. 5. An example of a directed network with a length 3 cycle and subnetworks with some links removed.

We prove that $C_{i.i.d.}(G) = C_{i.i.d.}(G_k)$ for at least one value of k, $k = 1, 2, \ldots, N$ by showing that each of the following assumptions lead to a contradiction:

(a) $C_{i.i.d.}(G) > C_{i.i.d.}(G_k)$ for $k = 1, 2, \ldots, N$;
(b) $C_{i.i.d.}(G) < C_{i.i.d.}(G_k)$ for at least one value of k, $C_{i.i.d.}(G) \neq C_{i.i.d.}(G_k)$ for $k = 1, 2, \ldots, N$.

Case (a): Assume $C_{i.i.d.}(G) > C_{i.i.d.}(G_k)$ for $k = 1, 2, \ldots, N$. Given our assumption, we first show that for each subnetwork G_k, there exists a cut S'_k that achieves the minimum cut value, i.e.,

$$C_{i.i.d.}(G_k) = f(G_k, S'_k) \tag{16}$$

with the property $v_1, v_2, \ldots, v_k \in S'_k$ and $v_{k+1} \notin S'_k$. This will lead to a contradiction when we take $k = N$.

If $C_{i.i.d.}(G) > C_{i.i.d.}(G_k)$, then $v_k \in S_k$ and $v_{k+1} \in \overline{S_k}$. This can be seen as follows. Any cut that does not cross the link (v_k, v_{k+1}) has the same cut value for both G and G_k by Property (1). So the minimum cut value attained by G_k must be for a cut which crosses the link (v_k, v_{k+1}) and yields a cut value strictly less than any cut which does not cross that link. Thus, for $k = 1$ we can choose $S'_1 = S_1$.

We will discover the sets S'_k for larger k by induction. We will show that if the claim in (16) holds for $k-1$, it should also hold for k.

First note that since S'_{k-1} and S_k achieve the minimum cut values for G_{k-1} and G_k, they must be less than or equal to any other cut in G_{k-1} and G_k respectively. In particular,

$$f(G_{k-1}, S'_{k-1}) \leq f(G_{k-1}, S'_{k-1}\cap S_k), \tag{17}$$
$$f(G_k, S_k) \leq f(G_k, S'_{k-1}\cup S_k). \tag{18}$$

Now, since $v_k \in \overline{S'_{k-1}}$ and $v_k \in \overline{S'_{k-1}\cap S_k}$, (v_k, v_{k+1}) cannot be an outgoing link in either of the cuts S'_{k-1} and $S'_{k-1}\cap S_k$, and all other links in G_{k-1} are also in $G_{k-1,k}$, so by Property (1) of f, we have

$$f(G_{k-1,k}, S'_{k-1}) = f(G_{k-1}, S'_{k-1}), \tag{19}$$
$$f(G_{k-1,k}, S'_{k-1}\cap S_k) = f(G_{k-1}, S'_{k-1}\cap S_k). \tag{20}$$

Also, $v_k \in S_k$ and $v_k \in S'_{k-1}\cup S_k$, so (v_{k-1}, v_k) cannot be an outgoing link in either of those cuts, and all other links in G_k are also in $G_{k-1,k}$. So again by Property (1) of f, we have

$$f(G_{k-1,k}, S_k) = f(G_k, S_k), \tag{21}$$
$$f(G_{k-1,k}, S'_{k-1}\cup S_k) = f(G_k, S'_{k-1}\cup S_k). \tag{22}$$

By the submodular property of f on $G_{k-1,k}$ and equations (19)-(22), we have:

$$f(G_{k-1}, S'_{k-1}) + f(G_k, S_k) \geq$$
$$f(G_{k-1}, S'_{k-1}\cap S_k) + f(G_k, S'_{k-1}\cup S_k).$$

Combining this result and equations (17) and (18) yields

$$f(G_k, S_k) = f(G_k, S'_{k-1}\cup S_k).$$

Thus the cut $S'_k = S'_{k-1}\cup S_k$ achieves the minimum cut value for network G_k and has the property $v_1, \ldots, v_k \in S'_k$. Now suppose $v_{k+1} \in S'_k$. Then the cut S'_k cannot cross the link (v_k, v_{k+1}), and thus $f(G_k, S'_k) = f(G, S'_k)$. But since S'_k achieves the minimum cut value for network G_k, we have the following:

$$\begin{aligned} C_{i.i.d}(G_k) &= f(G_k, S'_k) \\ &= f(G, S'_k) \\ &\geq C_{i.i.d}(G), \end{aligned}$$

which would contradict our assumption. The last inequality follows because $C_{i.i.d}(G)$ must be less than or equal to any cut value of G. Thus, $v_{k+1} \notin S'_k$. Letting $k = N$, $C_{i.i.d.}(G_N) =$

$f(G_N, S'_N)$. The link (v_N, v_1) is not an outgoing link in S'_N, and all other links in G_N are also in G, so by Property (1) of f,

$$C_{i.i.d.}(G_N) = f(G_N, S'_N) = f(G, S'_N) \geq C_{i.i.d.}(G),$$

which contradicts our assumption.

Case (b): Assume $C_{i.i.d.}(G) < C_{i.i.d.}(G_k)$ for at least one value of k, $C_{i.i.d.}(G) \neq C_{i.i.d.}(G_k)$ for $k = 1, 2, \ldots, N$.

Without loss of generality, let $C_{i.i.d.}(G) < C_{i.i.d.}(G_1)$. By the same arguments as in the previous case, $v_1 \in S^*$, $v_2 \in \overline{S^*}$.

We now show that for $k > 1$, if $v_k \in \overline{S^*}$, then $v_{k+1} \in \overline{S^*}$. This will lead to a contradiction when we take $k = N$.

Assume $v_k \in \overline{S^*}$ and consider $C_{i.i.d}(G_k)$:

$$\begin{aligned} C_{i.i.d}(G) &= f(G, S^*) \\ &\overset{(a)}{=} f(G_k, S^*) \\ &\overset{(b)}{>} C_{i.i.d}(G_k). \end{aligned}$$

(a) follows by Property (1) and the fact that $v_k \in \overline{S^*}$, and so S^* cannot cross the link (v_k, v_{k+1}). (b) follows from the fact that $C_{i.i.d}(G_k)$ must be less than or equal to any cut value of G_k and our assumption that $C_{i.i.d}(G) \neq C_{i.i.d}(G_k)$ for $k = 1, 2, \ldots, N$, thus making the inequality strict. Now since $C_{i.i.d}(G_k) < C_{i.i.d}(G)$, for the minimum cut S_k, we must have $v_k \in S_k$, $v_{k+1} \in \overline{S_k}$. Next, consider $S^* \cap S_k$. We have

$$f(G_k, S^* \cap S_k) \overset{(a)}{=} f(G, S^* \cap S_k) \tag{23}$$

$$\overset{(b)}{>} f(G_k, S_k), \tag{24}$$

where (a) follows by Property (1) and the fact that $v_k \in \overline{S^* \cap S_k}$, and so $S^* \cap S_k$ cannot cross the link (v_k, v_{k+1}). (b) follows because $C_{i.i.d}(G_k) < C_{i.i.d}(G)$, so the minimum cut value of $C_{i.i.d}(G_k) = f(G_k, S_k)$, must be strictly less than any cut value of G. Now suppose $v_{k+1} \in S^*$. Then

$$f(G_k, S^* \cup S_k) \overset{(a)}{=} f(G, S^* \cup S_k) \tag{25}$$

$$\overset{(b)}{\geq} f(G, S^*) \tag{26}$$

$$\overset{(c)}{=} f(G_k, S^*). \tag{27}$$

(a) and (c) follow because $v_{k+1} \in S^* \cup S_k$ and $v_{k+1} \in S^*$, so neither of those cuts can cross the link (v_k, v_{k+1}). (b) follows because S^* achieves the minimum cut value of graph G.

Combining (23)-(27), we have

$$f(G_k, S^* \cup S_k) + f(G_k, S^* \cap S_k) > f(G_k, S_k) + f(G_k, S^*).$$

This contradicts the submodularity of f in G_k. Thus $v_{k+1} \in \overline{S^*}$.

Letting $k = N$, we have $v_1 \in S^*, v_2, v_3, \ldots, v_N \in \overline{S^*}$. However, because $v_1, \ldots, v_N$ form a cycle, the node v_1 can be thought of v_{N+1} and $v_N \in \overline{S^*}$ by the above iteration implies that $v_1 \in \overline{S^*}$. This contradicts with the fact that $v_1 \in S^*$ and shows that the initial assumptions for case (b) necessarily lead to a contradiction.

Since we have eliminiated cases (a) and (b) above, we conclude that $C_{i.i.d}(G) = C_{i.i.d}(G_k)$ for at least one value of $k = 1, 2, \ldots, N$.

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